L_p-L_q Fourier multipliers on locally compact quantum groups

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Fourier transform and Fourier multipliers: $\mathbb{R}^n$

The Fourier transform $\widehat{f}$ of $f : \mathbb{R}^n \to \mathbb{C}$ is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

Given $\phi : \mathbb{R}^n \to \mathbb{C}$, the Fourier multiplier m_ϕ with symbol ϕ is

$$\widehat{m_\phi f}(\xi) = \phi(\xi) \widehat{f}(\xi), \ \ \xi \in \mathbb{R}^n,$$

or

$$(m_\phi \widehat{f})(\xi) = \widehat{\phi f}(\xi), \ \ \xi \in \mathbb{R}^n.$$

Questions: under which conditions of ϕ

- m_ϕ is bounded over $L_p(\mathbb{R}^n)$? (L_p-Fourier multiplier)
- m_ϕ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$? (L_p-L_q Fourier multiplier)

Hörmander's L_p-L_q Fourier multiplier theorem

Hörmander 1960 Acta. Math.

Let $1 < p \leq 2 \leq q < \infty$. Then for Fourier multiplier m_ϕ with symbol ϕ:

$$\|m_\phi : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)\| \precsim_{p,q} \sup_{s>0} s \left(\int_{\xi \in \mathbb{R}^n : |\phi(\xi)| \geq s} d\xi \right)^{\frac{1}{p} - \frac{1}{q}}.$$

The RHS is related to the weak L_r-spaces with $1/r = 1/p - 1/q$.

Weak L_p spaces and Lorentz spaces (1/2)

Fix a measure space (X, μ). The distribution function of a measurable function $f : X \to \mathbb{R}$ is

$$d_f(s) := \mu(\{x : |f(x)| > s\}).$$

The weak L_p-spaces, $0 < p < \infty$:

$$\|f\|_{p,\infty} := \sup_{s>0} s d_f(s)^{\frac{1}{p}}.$$

The more general Lorentz spaces, for $0 < p < \infty, 0 < q \leq \infty$,

$$\|f\|_{p,q} := p^{\frac{1}{q}} \left(\int_0^\infty \left[s d_f(s)^{\frac{1}{p}} \right]^q \frac{ds}{s} \right)^{\frac{1}{q}}, \quad q < \infty.$$

Weak L_p spaces and Lorentz spaces (2/2)

Basic properties of Lorentz spaces

- $L_{p,p} = L_p \subset L_{p,\infty}$.
- $L_{p,q} \subset L_{p,r} \subset L_{p,\infty}$ whenever $q < r < \infty$.
- for $1/p_0 = 1/p_1 + 1/p_2, 1/q_0 = 1/q_1 + 1/q_2$:

$$\|fg\|_{p_0,q_0} \precsim \|f\|_{p_1,q_1} \|g\|_{p_2,q_2}.$$

Some follow-ups of Hörmander’s theorem

- compact Lie groups [Akylzhanov, Nursultanov and Ruzhansky, 2016]
- locally compact separable unimodular groups [Akylzhanov and Ruzhansky, 2016]
- compact quantum groups of Kac type [Akylzhanov, Majid and Ruzhansky, 2018]
- ...

This talk: certain locally compact quantum groups, slightly simpler.

Main result

Let $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ be a locally compact quantum group with dual $\widehat{\mathbb{G}} = (\widehat{\mathcal{M}}, \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$.

Theorem Z. 2021

Let $1 < p \leq 2 \leq q < \infty$ and $1/r = 1/p - 1/q$. Suppose that φ and $\widehat{\varphi}$ are both tracial. Then the Fourier multiplier m_x with symbol x satisfies

$$\|m_x : L_p(\widehat{\mathbb{G}}, \widehat{\varphi}) \to L_q(\widehat{\mathbb{G}}, \widehat{\varphi})\| \precsim_{p,q} \|x\|_{L_{r,\infty}(\mathbb{G},\varphi)}.$$

Corollary

When $\widehat{\mathbb{G}}$ is a compact quantum group of Kac type, such as group von Neumann algebra $\widehat{G}$ of a discrete group G, then

$$\|m_x : L_p(\widehat{\mathbb{G}}, \widehat{\varphi}) \to L_p(\widehat{\mathbb{G}}, \widehat{\varphi})\| \precsim_{p} \|x\|_{L_{p^*,\infty}(\mathbb{G},\varphi)},$$

where $1 < p < \infty$ and $1/p^* = |1/2 - 1/p|$.

Plan

Simpler proof of $\mathbb{R}^n$ case

Noncommutative L_p-spaces, Lorentz spaces and interpolation

Locally compact quantum groups and Fourier transform

Proof of main results and remarks

Hausdorff–Young inequalities for $\mathbb{R}^n$

We know that the Fourier transform on $\mathbb{R}^n$ satisfies

$$\|\widehat{f}\|_\infty \leq \|f\|_1,$$

and Parseval–Plancherel identity

$$\|\widehat{f}\|_2 = \|f\|_2.$$

Then by complex interpolation, we get the famous

Hausdorff–Young inequalities

Let $1 < p < 2$ and $1/p' + 1/p = 1$. Then

$$\|\widehat{f}\|_{p'} \leq \|f\|_p.$$

Interpolation of L_p-spaces

For any $1 \le p_0, p_1, q_0, q_1 \le \infty$ and $0 \le \theta \le 1$. Put

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T is linear such that

$$\|T : L_{p_i} \to L_{q_i}\| := M_i < \infty, \quad i = 0, 1,$$

then

- Complex interpolation

$$\|T : L_{p_\theta} \to L_{q_\theta}\| \le M_0^{1-\theta} M_1^{\theta}.$$

- Real interpolation: $\forall r$

$$\|T : L_{p_\theta, r} \to L_{q_\theta, r}\| \lesssim M_0^{1-\theta} M_1^{\theta}.$$

Hausdorff–Young inequalities via interpolation

Consider the Fourier transform and couples (L_1, L_2) and (L_∞, L_2):

$$\|\widehat{f}\|_\infty \leq \|f\|_1, \ \ \|\widehat{f}\|_2 = \|f\|_2.$$

So for $1 < p < 2$ and $1/p' + 1/p = 1$, the complex interpolation gives

$$\|\widehat{f}\|_{p'} \leq \|f\|_p,$$

while the real interpolation gives real Hausdorff–Young inequalities

$$\|\widehat{f}\|_{p'} \leq c_p \|f\|_{p,p'}, \ \ \|\widehat{f}\|_{p',p} \leq c_p \|f\|_p.$$

The real version is stronger in the sense that: whenever $p < q < \infty$

$$L_p = L_{p,p} \subset L_{p,q} \subset L_{p,\infty}.$$

A simpler proof for Hörmander's theorem

We need to show that

$$\|m_\phi : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)\| \lesssim \|\phi\|_{r,\infty},$$

where $1 < p \le 2 \le q < \infty$ and

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'}.$$

The proof is as follows:

$$\begin{aligned}
\|\widehat{\phi f}\|_q &\lesssim \|\phi f\|_{q',q} && \text{real Hausdorff–Young} \\
&\lesssim \|\phi\|_{r,\infty} \|f\|_{p',q} && \text{Hölder for Lorentz spaces} \\
&\lesssim \|\phi\|_{r,\infty} \|f\|_{p',p} && \text{inclusoin of Lorentz spaces} \\
&\lesssim \|\phi\|_{r,\infty} \|\hat{f}\|_{p}. && \text{real Hausdorff–Young for dual}
\end{aligned}$$

Noncommutative L_p-spaces

Let $\mathcal{M}$ be a von Neumann algebra (i.e. a *-subalgebra of $B(H)$ that is weak-closed) equipped with a normal semifinite faithful trace τ.

Examples

- $(\mathcal{M}, \tau) = (L_\infty(X), \int \cdot d\mu)$ for a measure space (X, μ),
- $(\mathcal{M}, \tau) = (M_n(\mathbb{C}), \mathrm{Tr})$.

For $0 < p < \infty$, put

$$\|x\|_p^p := \tau|x|^p = \tau(x^*x)^{p/2}, \quad x \in \mathcal{M}.$$

Then $L_p(\mathcal{M})$ is the completion of $(\mathcal{M}, \|\cdot\|_p)$. Set

$$L_\infty(\mathcal{M}) := \mathcal{M}, \quad \|x\|_\infty := \|x\|.$$

Noncommutative Lorentz spaces

The NC distribution function:

$$\lambda_s(x) := \tau(1_{(s,\infty)}(|x|)), \quad s \geq 0.$$

Then one can define NC Lorentz spaces in a similar way, and

- basic properties of NC Lorentz spaces: ✓
- complex interpolation and real interpolation results: ✓

Remarks

If τ is a weight, i.e. $\tau(ab) \neq \tau(ba)$, then

- NC $L_p(\mathcal{M}, \tau)$ and complex interpolation: ✓
- NC $L_{p,q}(\mathcal{M}, \tau)$ and real interpolation: ✗, i.e. in general

$$(L_\infty, L_1)_{1/p} \neq (L_\infty, L_1)_{1/p,p}.$$

Locally compact groups

About a locally compact group (G, μ, ν):

- a topological group: group + topology, compatible
- examples: (locally compact, compact, discrete) group: $\mathbb{R}, \mathbb{T}, \mathbb{Z}$
- any locally compact group has a left Haar measure μ:

$$\mu(sE) = \mu(E), \forall s \in G, E \subset G; \quad \text{(left invariant)}$$

  and a right Haar measure ν:

$$\mu(Es) = \mu(E), \forall s \in G, E \subset G. \quad \text{(right invariant)}$$

- G is unimodular if $\mu = \nu$ (up to a factor). Examples: $(\mathbb{R}, dx)$.

What is a locally compact quantum group?

Motivation: non-abelian Pontryagin duality

Pontryagin duality

Let G be a locally compact abelian group. Its dual

$$\widehat{G} := \{\text{irreducible unitary strongly continuous representation of } G\}$$

is also a locally compact abelian group. We have $\widehat{\widehat{G}} \cong G$.

Examples

$$\begin{array}{lllll} G: & \mathbb{Z}_n & \mathbb{Z} & \mathbb{T} & \mathbb{R}^n \\ \widehat{G}: & \mathbb{Z}_n & \mathbb{T} & \mathbb{Z} & \mathbb{R}^n \end{array}$$

What if G is non-abelian? $\widehat{G}$ is NOT a group in general, but it is a locally compact quantum group and Pontryagin duality still holds.

Pontryagin duality: abelian

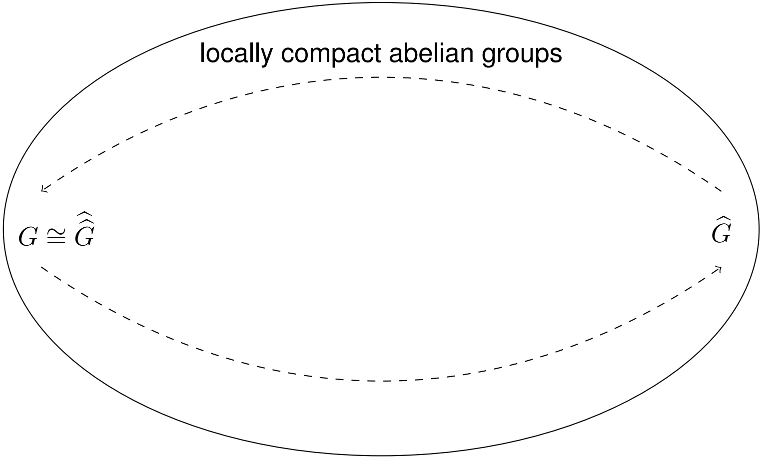

Figure: Pontryagin duality in the category of locally compact abelian groups

Pontryagin duality: abelian

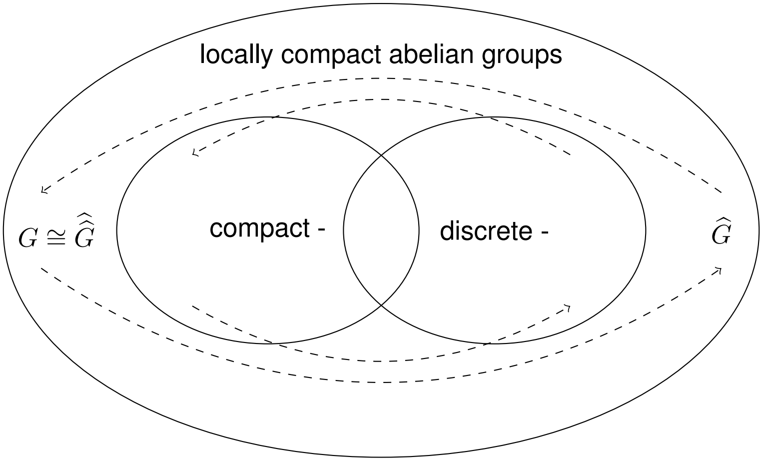

Figure: Pontryagin duality in the category of locally compact abelian groups

Pontryagin duality: non-abelian and quantum

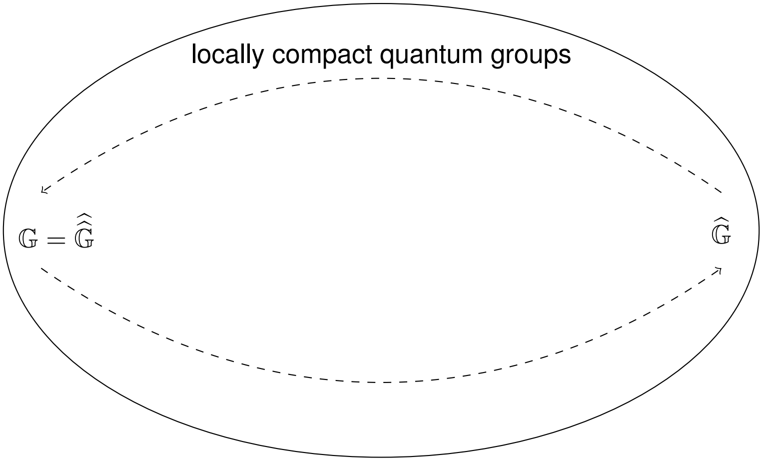

Figure: Pontryagin duality in the category of locally compact quantum groups

Pontryagin duality: non-abelian and quantum

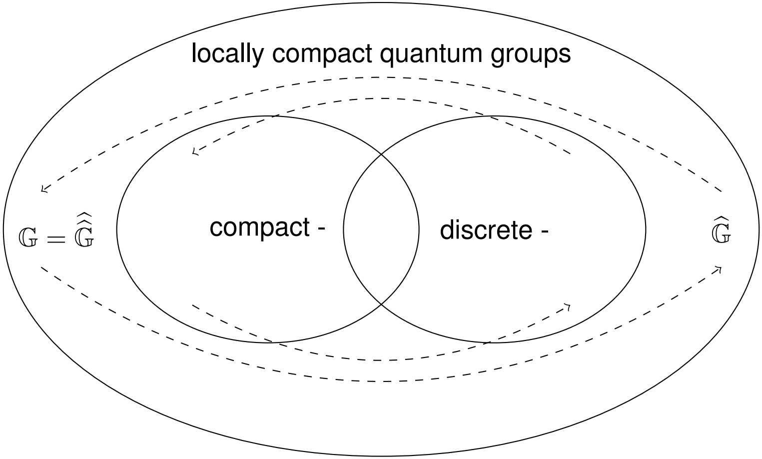

Figure: Pontryagin duality in the category of locally compact quantum groups

Locally compact quantum groups

Kustermans and Vaes, 2000, 2003

A locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ consists of

1. a von Neumann algebra $\mathcal{M}$; ($L^\infty(G)$)
2. a normal, unital, *-homomorphism $\Delta : \mathcal{M} \to \mathcal{M}\overline{\otimes}\mathcal{M}$ such that

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta; (G \times G \to G)$$

3. a normal, semifinite, faithful weight φ which is left invariant

$$\varphi[(\omega \otimes \text{id})\Delta(x)] = \varphi(x)\omega(1), \ \ \omega \in \mathcal{M}_*^+; (\text{left Haar measure } \mu)$$

4. a normal, semifinite, faithful weight ψ which is right invariant

$$\psi[(\text{id} \otimes \omega)\Delta(x)] = \psi(x)\omega(1), \ \ \omega \in \mathcal{M}_*^+. (\text{right Haar measure } \nu)$$

Fourier transform: abelian groups

When (G, μ) is locally compact abelian, the Fourier transform $\mathcal{F}$ is

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_G f(s)\overline{\xi(s)}d\mu(s), \ \ \xi \in \widehat{G}.$$

What if G is non-abelian? Observation: when G is abelian

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g), \ \ f \in L_1(G, \mu), \ g \in L_2(G, \mu).$$

So $\mathcal{F}(f)\cdot$ is unitarily equivalent to the convolution operator

$$\mathcal{F}\lambda(f) = \mathcal{F}(f) \cdot \mathcal{F}.$$

Fourier transform: non-abelian groups

The left regular representation λ of G:

$$\lambda_s g(t) = g(s^{-1}t), \;\; g \in L_2(G, \mu).$$

Kunze 1958

G unimodular and non-abelian. The Fourier transform:

$$\widehat{f} = \lambda(f) := \int_G f(s)\lambda_s d\mu(s) \in B(L_2(G)), \;\; f \in L_1(G).$$

We still have Hausdorff–Young inequalities:

$$\|\lambda(f)\|_{L_{p'}(\widehat{G})} \leq \|f\|_{L_p(G)}, \;\; 1 \leq p \leq 2.$$

further generalized to non-unimodular case (Terp 2017).

Fourier transform: quantum group case

Let $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ be a locally compact quantum group.

Theorem-Definition of (left) multiplicative unitary

There exists a unitary operator $W \in B(L_2(\mathbb{G}, \varphi) \otimes L_2(\mathbb{G}, \varphi))$ such that

$$W^*(x \otimes y) = \Delta(y)(x \otimes 1).$$

This operator contains all the data of $\mathbb{G}$. When $\mathbb{G} = G$, then

$$Wf(s,t) = f(s, s^{-1}t).$$

Then the Fourier transform of $\omega \in L_1(\mathbb{G}) = \mathcal{M}_*$:

$$\lambda(\omega) := (\omega \otimes \text{id})W.$$

Hausdorff–Young inequalities for $\mathbb{G}$

One can define the Fourier transform $\mathcal{F}$ on $L_p(\mathbb{G})$ and by complex interpolation we still have

Hausdorff–Young inequalities: Cooney '10, Caspers '13

Let $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ be a locally compact quantum group. Then

$$\|\mathcal{F}(x)\|_{L_{p'}(\widehat{\mathbb{G}},\widehat{\varphi})} \leq \|x\|_{L_p(\mathbb{G},\varphi)}, \quad 1 \leq p \leq 2.$$

Real Hausdorff–Young inequalities

Replace complex interpolation with real interpolation:

Real Hausdorff–Young inequalities

Let $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ be a locally compact quantum group with dual $\widehat{\mathbb{G}} = (\widehat{\mathcal{M}}, \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$. If φ and $\widehat{\varphi}$ are tracial, then

$$\|\mathcal{F}(x)\|_{p'} \lesssim_p \|x\|_{p,p'}, \quad \|\mathcal{F}(x)\|_{p',p} \lesssim_p \|x\|_p, \quad 1 \leq p \leq 2.$$

Proof of main result

The proof is similar to the $\mathbb{R}^n$ case:

$$\begin{aligned}
\|\mathcal{F}(xy)\|_q &\precsim \|xy\|_{q',q} && \text{real Hausdorff–Young for } \mathbb{G}\\
&\precsim \|x\|_{r,\infty}\|y\|_{p',q} && \text{Hölder for NC Lorentz spaces}\\
&\precsim \|x\|_{r,\infty}\|y\|_{p',p} && \text{inclusoin of NC Lorentz spaces}\\
&\precsim \|x\|_{r,\infty}\|\mathcal{F}(y)\|_p . && \text{real Hausdorff–Young for } \widehat{\mathbb{G}}
\end{aligned}$$

Remarks

- The index r for

$$\|m_x : L_p(\widehat{\mathbb{G}}, \widehat{\varphi}) \to L_q(\widehat{\mathbb{G}}, \widehat{\varphi})\| \precsim_{p,q} \|x\|_{L_{r,\infty}(\mathbb{G},\varphi)}.$$

  is sharp for $\mathbb{G} = G = \mathbb{T}$.
- When $\widehat{\mathbb{G}}$ is a compact quantum group of Kac type, the index p for

$$\|m_x : L_p(\widehat{\mathbb{G}}, \widehat{\varphi}) \to L_p(\widehat{\mathbb{G}}, \widehat{\varphi})\| \precsim_p \|x\|_{L_{p^*,\infty}(\mathbb{G},\varphi)},$$

  is sharp, i.e. it cannot hold for $p = 1$ and $\mathbb{G} = G = \mathbb{T}$.

Questions

- No reasonable Lorentz space $L_{p,q}(\mathcal{M},\tau)$ and real interpolation interpolation for general weight τ.
- However, for a non-unimodular group G, the following question still makes sense: do we have

$$\|m_\phi : L_p(\widehat{G},\widehat{\varphi}) \to L_q(\widehat{G},\widehat{\varphi})\| \precsim_{p,q} \|\phi\|_{L_{r,\infty}(G,\varphi)}?$$

  Here φ is tracial while $\widehat{\varphi}$ is not. Our proof doesn't work.

Questions

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$$\|m_\phi : L_p(\widehat{G}, \widehat{\varphi}) \to L_q(\widehat{G}, \widehat{\varphi})\| \precsim_{p,q} \|\phi\|_{L_{r,\infty}(G,\varphi)}?$$

Here φ is tracial while $\widehat{\varphi}$ is not. Our proof doesn't work.

Thank you!

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